ON THE FIELD ANOMALY OF NEAR WAKES IN A COLLISIONLESS PLASMA*

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A variational approach is proposed to determine the induced electric field by charge separation in the near wake of a large negatively charged body moving at mesothermal speeds in a tenuous plasma. The presence of potential well is discussed.

The interaction between the ionospheric medium and a rapidly moving body provides a unique plasma phenomenon of much geophysical consequences [1]. The moving body in question, e.g., a space probe or meteorite, can be charaterized by the following conditions: $\lambda_D \ll R \ll l; c_i \ll V \ll c_e$ where R and V denote the size and the speed respectively of the body; λ_D, l, c_i and c_e, the Debye length, mean free path, ion thermal speed and electron thermal speed respectively of the plasma. It is well known that when a neutral plasma meets a moving blunt body a potential sheath having a thickness of the order λ_D develops to shield the frontal body. In the near wake behind a rapidly moving body where the electrons lead the ions in filling the void region, a charge separation field in the form of a potential valley is created. In other words, the potential distribution in the near wake, instead of varying monotonically from its boundary value at the body surface ϕ_s (usually < 0) to zero at the free stream, dips into a well of depth $|\phi_{min}| > |\phi_s|$ before it rises to its ambient value (zero). This anomalous field configuration which has been shown to cause plasma trapping [2] and possibly other consequences yet to be explored is of interest in the present note.

Consider a conducting spherical body of radius R and surface potential ϕ_s (< 0) which moves at a steady mesothermal speed V in a bithermal neutral plasma of singly charged ions at temperature T_i and free electrons at temperature T_e where $\lambda_D \ll R \ll l$. It is assumed that in collision with the body an electron is absorbed; an ion, neutralized and re-emitted as a neutral particle. The magnetic field effect is neglected. These conditions hold e.g., for a typical satellite-motion in the upper ionosphere [3]. In the following analysis the origin of a cylindrical coordinate is fixed to the center of the moving body with its z-axis aligned with the axis of the wake. The linear displacement $\boldsymbol{r}$ is normalized by R; field potential $\phi(\boldsymbol{r})$ by e/kT_i where e and k denote the electronic charge and Boltzmann constant respectively; particle velocity $\boldsymbol{c}$ by c_i and the particle density $n(\boldsymbol{r})$ by n_∞, the free stream electron or ion density. The conditions $\phi_s < 0$ and $V \ll c_e$ imply that the electron distribution in the wake will deviate from its free stream value only by the Boltzmann factor thus $n_e(\boldsymbol{r}) = \exp(\phi\, T_i/T_e)$. On the other hand since $V \gg c_i$ the ion distribution $f(\boldsymbol{r}, \boldsymbol{c})$ in the wake will differ strongly with its free stream equilibrium state. In fact $f(\boldsymbol{r}, \boldsymbol{c})$ and $\phi(\boldsymbol{r})$ are governed by the following coupled equations of the collision free (particle) equation and the Poisson (field) equation [3]:

$$\boldsymbol{c}\cdot\nabla_r f - \tfrac{1}{2}(\nabla_r \phi)\cdot\nabla_c f = 0; \quad (\lambda_D/R)^2 \nabla_r^2 \phi = n_i - n_e \quad (1)$$

where $n_i = \int f \, d\boldsymbol{c}$ with $\phi = \phi_s$ at $r = R$ and $\phi = 0$ at $r \to \infty$.

An iterative numerical scheme has often been used [3] to treat the nonlinear system (1). It starts with a formal solution for $f(\boldsymbol{r}, \boldsymbol{c})$ from the quasi-linear first order particle equation, in terms of the appropriate invariants. Notice that an analogous mathematical formalism in stellar dynamics is known as Jeans theorem [4]. For an axi-symmetric field herein interested three isolating invariants are needed. The energy invariant $E = c_\rho^2 + c_\theta^2 + c_z^2 + \phi$ and angular momentum invariant $L_z = |\boldsymbol{r} \times \boldsymbol{c}|_z$ are obvious [3]; the existence of the third invariant for a general arbitrary axi-symmetric $\phi(\boldsymbol{r})$, however, has not been settled. Nonetheless, if the potential is prescribed as $\phi_L = \zeta(r) + \eta(\theta)/r^2$ where $\zeta(r)$ and

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$\eta(\theta)$ are arbitrary functions of the radius vector r and the polar angle θ respectively, it is known that the third invariant $I_L = |r \times c|^2 + \eta(\theta)$ [4].

It is suggested herein to use a variational method that yields approximate solutions to the system (1). It consists of finding a functional $J(\phi, f)$ whose variation gives the equations (1) as its "Euler equations" [5]. This is found to have the following form:

$$J(\phi,f) = \tfrac{1}{2}(\lambda_D/R)^2 \{\int (\nabla\phi)^2 \mathrm{d}r + \phi_s \int_{\text{sphere}} [\partial\phi(1,\theta)/\partial r]\, \mathrm{d}\theta\}$$
$$- \iint^{\phi} (n_i - n_e)\, \mathrm{d}\phi'\, \mathrm{d}r + \iint [c\cdot\nabla_r f + (\nabla_r\phi)\cdot\nabla_c f] f'\, \mathrm{d}c\, \mathrm{d}r \tag{2}$$

for an axi-symmetric wake. Trial functions for $\phi(r)$ and $f(r, c)$ must be furnished to functional (2) before a variational principle can be applied. It is intended to introduce a perturbed pair (ϕ, I_3) which is related to the unperturbed (ϕ_L, I_L)-pair as follows* [5]:

$$\phi = \phi_L + \eta_1(r,\theta)/r^2, \qquad I_3 = I_L + \eta_1(r,\theta) \tag{3}$$

I_3 will be used along with E and L_z to prescribe the trial $f(r, c) = f(E, L_z, I_3)$ following Jeans theorem. In compliance with (3), the suggested trial potential $\phi(r)$ which satisfies the wake boundary conditions is prescribed as follows:

$$\phi = \{\phi_s - (r-1)\cos^2\theta\, [B_0 p_0 + r\, B_1 p_1 \exp -(r-1)$$
$$+ r^2 B_2 p_2 \exp -2(r-1)$$
$$+ r^i B_i p_i \exp i(r-1) + \ldots]\} \exp -(r-1) \tag{4}$$

and its associated third invariant:

$$I_3 = r^2(c_\theta^2 + c_\phi^2) - r^2(r-1)\cos^2\theta\, [B_0 + \tag{5}$$
$$+ r B_1 p_1 \exp -(r-1) + r^2 B_2 p_2 \exp -2(r-1)$$
$$+ r^i B_i p_i \exp -i(r-1) + \ldots] \exp -(r-1)$$

where p_i $(i = 0, 1, 2, 3, \ldots)$ are Legendre polynomials; $B_i (i = 0, 1, 2, \ldots)$, coefficients to be determined with variational method. The trial $f(E, L_z, I_3)$ satisfying its boundary conditions [3] can be written

$$f = \tfrac{1}{2}\, \pi^{-3/2} [1 + \mathrm{sign}\,(S-1)] \exp[-(E - V^2)] \tag{6}$$

* A priori restriction thus implied is that ϕ deviates only slightly from ϕ_L.

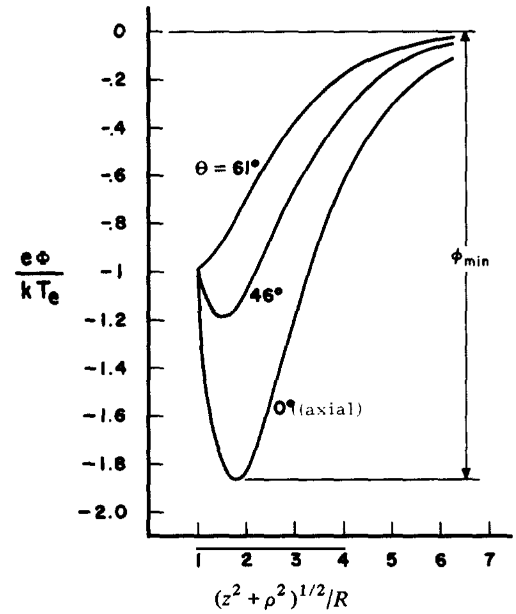

Fig. 1. Electric potential distribution in an axi-symmetric wake.

where $S = I_3/I_3$ (evaluated at $\theta = \pi/2$). The vanishing of the differential coefficients $\partial J/\partial B_i$ in the first variant δJ, after the substitution of eqs. (4), (5), and (6) in the functional (2), yields a set of nonlinear transcendental algebraic equations which can be solved by the generalized Newton's method. The result, $\phi(r)$, of a case with $\phi_s = -1$, $V/c_i = 8$, $\lambda_D/R = 0.05$ and $T_e = T_i$ is illustrated in fig. 1. In this illustrated variational calculation, the terms byond B_3 and p_3 in eqs. (4) and (5) have been truncated. Calculations including the B_4 and p_4 terms have also been made with, however, negligible change in results of $\phi(r)$ and $n_i(r)$. The present result (fig. 1) agrees well with previous numerical iteration results in values ϕ_{min} but shows a slight shift in ϕ-distribution [3]. The locus of the potential minimum (ϕ_{min}) forms a conical surface in an axi-symmetric wake. The presence of a potential valley in the wake stems from the charge separation as the ambient electrons and ions move into the wake with unequal mass motions.

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